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Dyson–Schwinger equations for the non-linear σ-model: perturbative solution on a finite lattice

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Abstract. The Dyson-Schwinger equations for the O(N)-symmetric non-linear σ-model to $\mathcal{O}(1/N)$ are solved perturbatively at weak coupling on a finite lattice. The solution equals the known second-order weak-coupling result to all powers in $1/N$ (third and higher powers have coefficient zero). In the Dyson-Schwinger equations zero-momentum modes cause no problems. On the contrary, their dominance is the basis for a simple expansion scheme.

1. Introduction

The O(N)-symmetric non-linear σ-model in two dimensions has several interesting properties in common with non-Abelian gauge theories in four dimensions. Its simpler action and fewer degrees of freedom make it a favourite testing ground for ideas ultimately intended for gauge theories. Recently, a subseries of its $1/N$ expansion was summed by means of Dyson-Schwinger equations [1]. These equations were solved numerically on square lattices for $N=3$. The numerical solutions for the magnetic susceptibility and for the mass gap were found to agree surprisingly well with Monte Carlo results for these quantities. This indicates that little of the model's full $1/N$ series is missed by the subseries studied. But these were numerical results. An analytical understanding is desirable, but requires other analytical expressions to compare with, expressions that describe the model as well, at least, as the Dyson-Schwinger equations were found to do, or the comparison is pointless. So there are few analytic results to choose from. The full $1/N$ series of the model is not known beyond the order 1, which is reproduced exactly by the Dyson-Schwinger equations. The strong-coupling expansion of the two-point function is a natural candidate at strong coupling, and one easily convinces oneself that its first few terms are reproduced by a hopping expansion of the Dyson-Schwinger equations. At weak coupling the weak-coupling expansion on a finite lattice is known to describe Monte Carlo results well [1]. In the present paper we solve the Dyson-Schwinger equations analytically at weak coupling on a finite lattice, and compare the result with the exact weak-coupling series for the two-point function, which is known to second order [2].

We find that the Dyson-Schwinger equations yield a two-point function identical to the exact weak-coupling result to the order it is known. This exact identity is probably an artefact of the particular comparison made. Unfortunately higher orders and other n-point functions in a finite volume are not available for comparison. Just the same, hitting the bullseye can hardly *discredit* a method, even if a little luck may have been involved.

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With the hindsight of this result, it is interesting to compare the two ways in which this two-point function can be found; especially because the differences hold for any n-point function.

The one-loop calculation of [2] required a rather delicate treatment of zero modes. The usual infrared regularisation by a small external magnetic field could not be used because, as was shown, the two limits *external field* $\to 0$ and *coupling* $\to 0$ do not commute for a finite lattice in two dimensions. (Neither do they for lattices of any size in one dimension [3]). Instead collective coordinates were introduced for the $N-1$ zero modes of the model. The Faddeev–Popov trick was used to handle these coordinates and led to the usual Feynman rules plus some extra vertices and the rule to leave out zero-momentum modes in Fourier sums in Feynman graphs.

An earlier treatment of zero modes used the global symmetry to fix the direction of one spin on the lattice. One finds the usual Feynman rules, with no extra vertices, but the propagator is no longer translation invariant. Instead it is a sum of four of the usual, infrared divergent propagators, with the position of the fixed spin as argument of three of these propagators [4]. This approach appears to be as complicated as that of [2].

In contrast to these approaches, the Dyson–Schwinger equations solved in the present paper are translation invariant and general. They have the same form in finite and infinite volumes, and zero modes cause no problem. On the contrary, their dominance in a finite volume at weak coupling is the basis for the simple expansion scheme used here.

2. Finite-volume weak-coupling expansion

The Dyson–Schwinger equations for the O(N)-symmetric non-linear σ-model in d dimensions have been derived to $\mathcal{O}(1/N)$ in [1]. They read

$$\tilde{G}_p^{-1} = \tilde{G}_p^{(0)-1} + \frac{1}{N}((\tilde{G} * \tilde{D})_p - (\tilde{G} * \tilde{D})_{p=0}) \tag{1}$$

$$\tilde{D}_p^{-1} = \tfrac{1}{2}(\tilde{G} * \tilde{G})_p \tag{2}$$

where '*' means convolution, and $\tilde{G}_p^{(0)}$ is the bare propagator. In (1) and (2) terms explicitly of order two or more in $1/N$ have been neglected. This is why there are only two, fairly simple equations. However, since the equations are not homogeneous in $1/N$, their solution $(\tilde{G}, \tilde{D})$ depends on $1/N$ through all non-negative integer powers. This dependence is exact to orders $(1/N)^0$ and $(1/N)^1$ because (1) and (2) are. Nothing is known about higher orders except that a subseries of the $1/N$ expansion of the model is in effect summed by (1) and (2), and that little is missed by this sum according to the numerical result of reference [1]. As explained in the introduction, the purpose of this paper is to obtain an analytical solution of (1) and (2) in the limit of weak coupling and finite volume.

For the standard action on a square or cubic lattice with periodic boundary conditions $\tilde{G}_p^{(0)}$ is given by

$$\tilde{G}_p^{(0)-1} \equiv \omega + \tilde{V}_p \tag{3}$$

$$\tilde{V}_p \equiv 4 \sum_{\mu=1}^{d} \sin^2\left(\frac{p_\mu}{2}\right) \tag{4}$$

where d is the dimension of the lattice. $\tilde{G}_p$ and $\tilde{D}_p$ are the Fourier transforms of G_x and D_x. ω is a control parameter in this set of equations. The inverse coupling is a function of ω

$$\lambda \equiv \frac{\beta}{N} = G_{x=0} = L^{-d} \sum_p \tilde{G}_p \tag{5}$$

and so is the magnetic susceptibility

$$\chi_m = \frac{\sum_x G_x}{G_{x=0}} = \frac{\tilde{G}_{p=0}}{G_{x=0}} = \frac{1}{\omega\lambda}. \tag{6}$$

Here L is the linear extent of the lattice, L^d being the number of sites on the lattice.

Where the infinite-volume system has a critical point, the finite-volume system has $\chi_m \sim L^{d-\eta}$, where $\eta/2$ is the anomalous dimension. Thus from (6), rewritten as

$$\omega = \frac{1}{\lambda\chi_m} \tag{7}$$

we see that ω is small for $\lambda \sim \lambda_{\text{critical}}$, both for $\lambda_{\text{critical}}$ finite and, of course, for $\lambda_{\text{critical}}$ infinite. Consequently one may expand in ω. We do this under the assumption that

$$\tilde{G}_{p=0} = \frac{1}{\omega} \gg \tilde{G}_{p\neq 0}. \tag{8}$$

Then we can easily solve (1)-(4) for $\tilde{G}_p$ expanded in powers of ω. From the solution for $\tilde{G}_p$ below we shall see that (8) is indeed satisfied for small values of ω, i.e. the assumption is self-consistent.

Once we have found $\tilde{G}_p$ in powers of ω, we may eliminate ω in favour of λ. This is done by using (10) below. Equation (8) used in (5) gives

$$\lambda = L^{-d}\left(\frac{1}{\omega} + \sum_{p\neq 0} \tilde{G}_p\right). \tag{9}$$

Using $\tilde{G}_{p\neq 0} \sim \mathcal{O}(1)$, equation (9) is inverted to

$$\omega L^d = \lambda^{-1} + \lambda^{-2} L^{-d} \sum_{p\neq 0} \tilde{G}_p + \mathcal{O}(\lambda^{-3}). \tag{10}$$

Notice that expansion in ω to $\mathcal{O}(\omega)$ may be a good approximation, and (10) to $\mathcal{O}(\lambda^{-1})$ at the same time a considerably worse approximation.

The expansion in powers of ω is done as follows: using (8) we rewrite (2) as

$$\tilde{D}^{-1}_{p=0} = \frac{N}{2L^d}\left(\frac{1}{\omega^2} + \sum_{q\neq 0} \tilde{G}_q\right) \tag{11}$$

$$\tilde{D}^{-1}_{p\neq 0} = \frac{N}{2L^d}\left(\frac{2}{\omega}\tilde{G}_p + \sum_{q\neq 0,p} \tilde{G}_{p-q}\tilde{G}_q\right) \tag{12}$$

and find

$$\tilde{D}_p = \delta_{p,0}\frac{2L^d}{N}\omega^2 + (1-\delta_{p,0})\frac{2L^d}{N}\left(\tfrac{1}{2}\omega\tilde{G}_p^{-1} - \tfrac{1}{4}\omega^2\tilde{G}_p^{-2}\sum_{q\neq 0,p}\tilde{G}_{p-q}\tilde{G}_q\right) + \mathcal{O}(\omega^3). \tag{13}$$

Using (13) we eliminate $\tilde{D}$ from (1) and (2), and (1) now becomes

$$\tilde{G}_p^{-1} = \tilde{G}_p^{(0)-1} + \frac{1}{N}(\tilde{G}_p^{-1} - \omega) + \frac{2\omega}{N}(\omega\tilde{G}_p - 1) + \frac{\omega}{N}\left(\sum_{q\neq 0,p}\tilde{G}^{-1}_{p-q}\tilde{G}_q - L^d + 2\right)$$
$$-\frac{\omega}{2N}(\tilde{G}_p^{-2} - \omega^2)\sum_{q\neq 0,p}\tilde{G}_{p-q}\tilde{G}_q + \mathcal{O}(\omega^2). \tag{14}$$

In (14) we have kept some explicit terms of $\mathcal{O}(\omega^2)$ to manifest that the RHS for $p=0$ equals $G_{p=0}^{(0)-1}=\omega$. Equation (14) can be rearranged to

$$\tilde{G}_p^{-1}=\frac{N}{N-1}\tilde{V}_p+\omega+\frac{2\omega}{N-1}(\omega\tilde{G}_p-1)+\frac{\omega}{N-1}\left(\sum_{q\neq 0,p}\tilde{G}_{p-q}^{-1}\tilde{G}_q-L^d+2\right)$$
$$-\frac{\omega}{2N}(\tilde{G}_p^{-2}-\omega^2)\sum_{q\neq 0,p}\tilde{G}_{p-q}\tilde{G}_q+\mathcal{O}(\omega^2). \tag{15}$$

For $p\neq 0$, equation (15) is solved for $\tilde{G}_p^{-1}$ to $\mathcal{O}(\omega)$ simply by replacing $\tilde{G}_p$ with $\tilde{V}_p^{-1}$ on the RHS and dropping terms of $\mathcal{O}(\omega^2)$. Using the explicit expression (4) for $\tilde{V}_p$ one finds

$$\tilde{G}_p^{-1}=\frac{N-1}{N}\omega+\left[\frac{N}{N-1}+\frac{\omega L^d}{N}\left(S_1-\frac{1}{2d}(1-L^{-d})\right)\right]\tilde{V}_p$$
$$-\frac{\omega}{2N}\left(\sum_{q\neq 0,p}\tilde{V}_{p-q}^{-1}\tilde{V}_q^{-1}\right)\tilde{V}_p^2+\mathcal{O}(\omega^2)\qquad\text{for } p\neq 0 \tag{16}$$
$$\tilde{G}_p^{-1}=\omega\qquad\text{for } p=0$$

where

$$S_n\equiv L^{-d}\sum_{q\neq 0}\tilde{V}_q^{-n}. \tag{17}$$

From (16) we see that the assumption (8) under which (16) was derived, is satisfied when

$$\omega\ll\frac{N}{N-1}\left(\frac{2\pi}{L}\right)^2+\mathcal{O}(\omega). \tag{18}$$

Equation (8) is an assumption that the size L of the system is small compared with the correlation length of the same system with infinite size L. Consequently there are no large distances nor small momenta (except zero) in the system, and it makes no sense to think about the mass gap or correlation length of $\tilde{G}_p$. This is manifest in (16) as a discontinuity in $\tilde{G}$ at $p=0$.

From (16) it follows that

$$\frac{N}{N-1}\tilde{G}_p=\left[1+\frac{\omega L^d}{N}\left(\frac{1}{2d}\left(1-\frac{1}{L^d}\right)-S_1\right)\right]\tilde{V}_p^{-1}$$
$$+\frac{\omega}{2N}\sum_{q\neq 0,p}\tilde{V}_{p-q}^{-1}\tilde{V}_q^{-1}-\frac{N-2}{N}\omega\tilde{V}_p^{-2}+\mathcal{O}(\omega^2)$$
$$=\left[1+\frac{1}{2d\beta}\left(1-\frac{1}{L^d}\right)-\frac{S_1}{\beta}\right]\tilde{V}_p^{-1}$$
$$+\frac{1}{2\beta}L^{-d}\sum_{q\neq 0,p}\tilde{V}_{p-q}^{-1}\tilde{V}_q^{-1}-\frac{N-2}{\beta L^d}\tilde{V}_p^{-2}+\mathcal{O}\left(\frac{1}{\beta^2}\right) \tag{19}$$

where we have used (10) to leading order and (5) to get the last identity.

3. Correlation function, and magnetic susceptibility

The spin-spin correlation function for the lattice O(N)-symmetric σ-model is related to the propagator G_x by

$$\langle S_x\cdot S_y\rangle=G_{x-y}/G_0. \tag{20}$$

From (5) it follows that

$$
\begin{aligned}
\langle \boldsymbol{S}_x \cdot \boldsymbol{S}_y \rangle &= 1 + \frac{1}{\lambda}(G_{x-y} - G_0) \\
&= 1 + \frac{N-1}{\beta} L^{-d} \sum_p (\mathrm{e}^{\mathrm{i}(x-y)\cdot p} - 1)\tilde{G}_p \\
&= 1 + \frac{N-1}{\beta}\left[1 + \frac{1}{2d\beta}\left(1 - \frac{1}{L^d}\right)\right](G^{\mathrm{P}}_{x-y} - G^{\mathrm{P}}_0) + \frac{N-1}{2\beta^2}(G^{\mathrm{P}}_{x-y} - G^{\mathrm{P}}_0)^2 \\
&\quad - \frac{(N-1)(N-2)}{\beta^2 L^d} \sum_z (G^{\mathrm{P}}_{x-z} G^{\mathrm{P}}_{z-y} - G^{\mathrm{P}}_{-z} G^{\mathrm{P}}_z) + \mathcal{O}\left(\frac{1}{\beta^3}\right)
\end{aligned}
\tag{21}
$$

where we have introduced the notation G^{P}_x for the massless propagator of perturbation theory:

$$
G^{\mathrm{P}}_x - G^{\mathrm{P}}_0 \equiv L^{-d} \sum_p (\mathrm{e}^{\mathrm{i}x\cdot p} - 1)\tilde{V}_p^{-1}. \tag{22}
$$

The sum in the last term on the RHS of (21) may also be written $L^{-d}\sum_p (\mathrm{e}^{\mathrm{i}(x-y)\cdot p} - 1)\tilde{V}_p^{-2}$. Equation (21) is identical to (13) in [2].

For the magnetic susceptibility (21) gives

$$
\begin{aligned}
\chi_m &= \sum_x \langle \boldsymbol{S}_x \cdot \boldsymbol{S}_0 \rangle \\
&= L^d \left(1 - \frac{1}{\lambda} L^{-d} \sum_{p \neq 0} \tilde{G}_p\right) \\
&= L^d \left[1 - \frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right)\right]
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
c_1 &= (N-1)S_1 \\
c_2 &= (N-1)\left[\tfrac{1}{2}S_1^2 - \frac{1}{2d} S_1 \left(1 - \frac{1}{L^d}\right)\right] + (N-1)(N-2)\frac{1}{L^d} S_2
\end{aligned}
\tag{24}
$$

with S_1 and S_2 as defined in (17). Table 1 lists values for S_1, S_2, and the expressions in which they occur in (23), for some two-dimensional square lattices. Notice that for $d = 2$ the infrared contribution to (17) makes S_1 grow with L as $\log L$ and S_2 as L^2. Consequently, for $d = 2$ the term proportional to S_2/L^d in (24) has a finite value for $L \to \infty$, and contributes a non-negligible amount to c_2 for L as in the table. This term, and the term proportional to $(N-1)(N-2)$ in (21), are not found if the usual infrared regularisation by a small external magnetic field is employed [2].

Table 1. The constants of (17) and (24) for some two-dimensional square lattices.

L	S_1	S_2	$\frac{1}{2}S_1^2 - \frac{1}{4}S_1(1-(1/L^2))$	$(1/L^2)S_2$
50	0.6714	9.75	0.0576	0.00390
100	0.7817	38.76	0.1101	0.00388
200	0.8920	154.79	0.1748	0.00387
400	1.0023	618.83	0.2518	0.00387

Let us finally compare results from the ordinary large-N expansion to order $1/N$ with our results from the Dyson–Schwinger equations to order $1/N$. Consider the spin–spin correlation function and magnetic susceptibility as they are obtained from the ordinary large-N expansion to order $1/N$ [5, 6]. Suppose we do a weak-coupling expansion to second order of these quantities. The results would contain terms only of $\mathcal{O}(1)$ and $\mathcal{O}(1/N)$, and therefore reproduce neither the third nor the fourth term on the RHS of (21), nor c_2/β^2 in (23) and (24), since these terms also contain N to the power -2.

In conclusion, we have seen that Dyson–Schwinger equations which are correct only to $\mathcal{O}(1/N)$, give a two-point function which at weak coupling is identical to all orders in $1/N$ with the exact second-order weak-coupling result. We have also seen that zero-momentum modes, far from causing problems, can be used as the basis for a weak coupling expansion scheme.

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